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Statistics of Close Visits to the Indifferent Fixed Point of an Interval Map

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We study a dynamical system defined by a map of the interval [0, 1] which has 0 as an indifferent fixed point but is otherwise expanding. We prove that the sequence of successive entrance times in a small neighborhood $[0, a]$ converges in law when suitably normalized to a homogeneous Poisson point process.

KEY WORDS: Poisson process; decay of correlations; intermittency; indifferent fixed point; map of the entrance time.

1. INTRODUCTION

In this paper we study a dynamical system defined by a map of the interval [0, 1] which has 0 as an indifferent (marginal) fixed point but is otherwise expanding. We allow the system to start with any absolutely continuous initial distribution with density of bounded variation whose support is away from 0.

We prove that the sequence of successive entrance times in a small neighborhood $[0, a]$ converges in law (in distribution) when suitably normalized to a homogeneous Poisson point process. The normalization does not depend on the initial distribution.

The prototype of this family of maps is the transformation

$$f(x) = \begin{cases} x/(1-x) & \text{for} \quad x \leqslant 1/2 \\ 2x-1 & \text{for} \quad x > 1/2 \end{cases}$$

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Systems of this type were suggested as models of temporal intermittence by Pomeau and Manneville.[14] One can distinguish a laminar regime when the point x is near the indifferent (marginal) fixed point ($x=0$) and a turbulent regime when the point x is away from this fixed point. In the laminar regime the time evolution is rather regular and slow, while in the turbulent regime we have instability and a sensitive dependence to initial conditions. The global time evolution of a typical trajectory is composed of laminar phases separated by turbulent bursts.

The main point about such systems is that although the SRB measure is the Dirac measure at the indifferent fixed point $x=0$,[11] the dynamics displays this interesting intermittence phenomenon. In other words, for any continuous function g and almost every initial condition x with respect to the Lebesgue measure, the asymptotic time average is $g(0)$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(f^n(x)) = g(0)$$

The turbulent bursts, on the other hand, are associated with another absolutely continuous invariant measure μ which is nonnormalizable.[2,4,10] If the function $g \in C^0$ has a support which does not contain the indifferent fixed point $x=0$, we have

$$\frac{\log n}{n} \sum_{j=0}^{n-1} g(f^n(x)) \xrightarrow[n\to\infty]{L^1} \int g\, d\mu$$

This behavior is related to the divergence of the expectation of the return times of the orbit, and to the poor mixing properties of this system. This is the main technical difficulty of the present paper.

In this picture our result says that the starting points of long laminar phases are Poissonian distributed. This can be illustrated as follows. Fix a small number $a>0$, and on a (discrete) tike (half) line put a dot at the instants n where the state x_n of the system is in the interval $]0, a]$. For a initial condition distributed according to the normalized Lebesgue measure on a closed interval which does not contain zero, one generates by this procedure a point process on the real half-line. Our main result states that one can find a normalization β_a (diverging when $a \searrow 0$) of the time scale such that the renormalized point process converges in law (in distribution) to a Poisson point process.

Let us discuss briefly the physical implications of this result. Since the statistics of Poisson point processes are so peculiar, this precise information should provide a sensitive test of the adequacy of the model. Moreover, the fact that the entrance times have independent exponential laws is a rigorous way of expressing the unpredictability of these events.

The fact that Poisson point processes appear as asymptotic laws for a broad class of dynamical situations seems to have attracted a lot of fishermen recently. As far as we know, the first result in this context is due to Doeblin.[7] In the context of finite Markov chains, the first result is due to Bellman and Harris[3] (see also ref. 13 and the interesting book by Aldous[1]). In the context of Axiom A dynamical systems the question was treated recently in several papers by Sinai,[16] Hirata,[8] and ourselves.[5] The present paper extends previous results obtained in ref. 6 for piecewise affine maps.

Even though the techniques are quite different, our method is similar to the one developed in the so-called pathwise approach to metastability and we refer to the survey paper by Schonmann[15] for a recent review of this subject.

The paper is organized in the following way. The class of dynamical systems we study is defined in the next section, which also contains the statement of the Main Theorem. Section 3 introduces the induced map which is the main tool of our proof and recalls some basic related results. Section 4 presents the basic *a priori* estimates about the first entrance time. Finally, Sections 5 and 6 develop the proof of the Main Theorem.

2. MODEL AND MAIN RESULT

Let f be the map of the unit interval $[0, 1]$ defined in the folowing way:

(i) $f(0) = 0$.

(ii) f is monotone C^3 on the intervals $]1/2, 1[$ and $]0, 1/2[$ and satisfies $f(]0, 1/2[) = f(]1/2, 1[) =]0, 1[$.

(iii) The slope of f is larger than one on $]0, 1/2[$ and larger (in absolute value) than a positive number $\bar{\rho} > 1$ on $]1/2, 1[$. Moreover, $f'(0) = 1$ and $f''(0) > 0$.

Note that it is easy to extend the map on each closed interval $[1/2, 1]$ and $[0, 1/2]$.

The interation of the map f defines a dynamical system system of the interval except for the countable set of preimages of the discontinuity point 1/2. We will only consider below absolutely continuous probability measures for the distribution of the initial condition. We will denote as usual by f^n the nth iterate of the map f.

Let $I = (x_n)$ be the decreasing sequence of preimages of 1/2 defined recursively by $x_0 = 1/2$ and for $n > 0$

$$x_n = \inf\{x \mid f(x) = x_{n-1}\}$$

We also define an integer-valued function U by

$$U(x)=\begin{cases}1 & \text{if } f(x)\in\,]1/2, 1] \\ n+1 & \text{if } f(x)\in\,]x_n, x_{n-1}[\end{cases}$$

We will study the sequence of entrance times into a small neighborhood $[0, a]$ of the indifferent fixed point $x=0$. More precisely, for any x belonging to $[1/2, 1]$ we define

$$T_a^1(x)=\inf\{n>0: f^n(x)\in[0, a]\}$$

and the sequence $T_a^j(x)$, for $j=2, 3,...,$ by

$$T_a^j(x)=\inf\{n>T_a^{j-1}(x)+U\circ f^{T_a^{j-1}(x)-1}(x)-1: f^n(x)\in[0, a]\}$$

In this definition we have adopted the usual convention which attributes the value $+\infty$ to the infimum of the empty set.

Let g be a positive function with bounded variation defined on the interval $[1/2, 1]$ with integral 1. We will consider the probability space $([1/2, 1], \mathscr{B}, g\,dx)$, where $\mathscr{B}$ denotes the Borel σ-field. The entrance times defined above are random variables in this probability space. From now on we will always choose the initial condition according to the probability measure $g(x)\,dx$.

From now on we will only take limits when a goes to 0 in the set I. This restriction will avoid boring and inessential difficulties in the text. The extension of the results when this restriction is eliminated is done in a standard way.

Main Theorem. There is a positive function β_a which does not depend on g such that the sequence of normalized stopping times $T^j_a\beta_a^{-1}$ converges in law to a mean-one homogeneous Poisson point process on $\mathbb{R}^+$ when $a\searrow 0$ in the set I.

Theorem 5 below will give a more precise estimate for the number β_a.

In the statement of thne above theorem and in what follows when no confusion is possible we will not mention the argument of the random variables. From now on we will most of the time not mention the fact that the limit $a\searrow 0$ is taken in the set I.

For definitions and fundamental results on Poisson point processes we refer to ref. 12.

3. THE INDUCED MAP

The main tool in the proof of the Theorem is the so-called induced map (on the interval $]1/2, 1]$), which is defined as follows. First of all we observe that the integer-valued function U defined above satisfies

$$U(x)=\inf\{n>0: f^n(x)\in\,]1/2, 1]\}$$

This is the number of steps the trajectory starting form x in $]1/2, 1]$ takes to return to this interval. The induced map F on the interval $]1/2, 1]$ is now defined by

$$F(x) = f^{U(x)}(x)$$

This induced map turns out to have much better expanding properties than f. In the following propositions we summarize those properties which will be used in the proof of the main theorem.

Proposition 1.

(1.1) There exists a decreasing sequence $(I_k)_{k \in \mathbb{N}}$ of closed intervals whose interiors are disjoint, whose union is the interval $]1/2, 1]$, and for every integer k, $F(I_k) = [1/2, 1]$.

(1.2) F is C^3 on each closed interval I_k, and there is a number $\rho > 1$ such that for any k

$$|F'_{I_k}| > \rho$$

We refer to ref. 2 for a proof of this proposition.

We will denote by $\mathscr{A}$ the partition of the interval $[1/2, 1]$ defined by the sequence I_k. The result (1.1) means that the partition $\mathscr{A}$ is Markov. We will denote by $\mathscr{A}_k$ the refined partition

$$\mathscr{A}_k = \bigvee_{l=0}^{k} F^{-l}\mathscr{A}$$

Proposition 2.

(2.1) F has a unique invariant ergodic, absolutely continuous probability measure μ. Moreover, the logarithm of its density h has bounded variation.

(2.2) The dynamical system defined by F and μ is uniformly mixing. Moreover, the following stronger property holds. There is a positive constant C and a positive number $\theta < 1$ such that for any positive integers k and n and for any measurable set B

$$\sup_{A \in \mathscr{A}_k} \left| \frac{\mu(A \cap F^{-n-k-1}(B))}{\mu(A)} - \mu(B) \right| \leqslant C\theta^n \mu(B)$$

Proof. We refer to ref. 17 for a proof of (2.1). For (2.2), we first observe that the map F has the following property of decay of correlations. There is a positive constant C_1 and a positive number $\theta < 1$ such that for any function g_1 of bounded variation and any integrable function g_2, we have for any integer m

$$\left|\int g_1 g_2 \circ F^m \, d\mu - \int g_1 \, d\mu \int g_2 \, d\mu\right|$$
$$\leqslant C_1 \theta^m \int |g_2| \, d\mu \left[\operatorname{Var}(g_1) + \int |g_1| \, d\mu\right] \tag{D.C.}$$

where $\operatorname{Var}(g_1)$ is the variation of the function g_1. This is a standard consequence of the material contained in ref. 17 using the techniques of ref. 9.

Let A be a atom of the partition $\mathscr{A}_k$ and B a measurable set. We can write

$$\mu(A \cap F^{-n-k-1}(B)) = \int \chi_A \chi_B \circ F^{n+k+1} h \, dx$$

Since h is bounded away from zero by (2.1), by the elementary properties of the transfer operator this is equal to

$$\int \frac{1}{h} P^{k+1}(\chi_A h) \chi_B \circ F^n h \, dx$$

If we take g_1 as $P^{k+1}(\chi_A h)/h$ and $g_2 = \chi_B$, we can apply formula (D.C.) to get

$$\left|\frac{\mu(A \cap F^{-n-k-1}(B))}{\mu(A)} - \mu(B)\right| \leqslant C\theta^n \mu(B)[\operatorname{Var}(g_1) + \mu(A)]$$

We now have to get an upper bound for $\operatorname{Var}(g_1)$. Since A is an atom of the partition $\mathscr{A}_k$, F^{k+1} is a diffeomorphism from A to $[1/2, 1]$. Call ψ its inverse, which is a diffeomorphism from $[1/2, 1]$ to A. It is easy to verify that

$$g_1(x) = \frac{h \circ \psi(x)}{|F^{k+1'} \circ \psi(x)| \, h(x)}$$

It is now straightforward to estimate the variation of g_1, and the proposition follows using the bound

$$\sup_{x \in [1/2, 1]} |F^{k+1'} \circ \psi(x)|^{-1} \leqslant \mathcal{O}(1) \, \mu(A)$$

From now on we will denote by $\mathscr{F}_k$ the σ-algebra generated by the partition $\mathscr{A}_k$. We remark that the above proposition also holds if we replace the atoms of $\mathscr{A}_k$ by the measurable sets in $\mathscr{F}_k$.

We recall that a positive integer-valued function τ is called a stopped time with respect to the sequence of σ-algebras $\mathscr{F}_k$, $k=0, 1,...,$ if the set $\{\tau=j\}$ belongs to $\mathscr{F}_j$ for all nonnegative integer j. As usual we will denote by $\mathscr{F}_\tau$ the σ-algebra of all the measurable sets B such that $B\cap\{\tau=j\}$ belongs to $\mathscr{F}_j$ for any nonnegative integer j.

The next corollary extends the mixing property (2.2) to stopping times.

Corollary 3. Let τ be a stopping time with respect to the sequence of σ-algebras $(\mathscr{F}_k)$. Then for any measurable set B and for any integer n we have

$$\sup_{A\in\mathscr{F}_\tau}\left|\frac{\mu(A\cap F^{-\tau-n-1}(B))}{\mu(A)}-\mu(B)\right|\leqslant C\theta^n\mu(B)$$

Proof. Let (D_k) be the partition associated to the stopping time τ and defined by

$$D_k=\{\tau=k\}$$

For any set A belonging to $\mathscr{F}_\tau$ we have by definition

$$\mu(A\cap F^{-\tau-n-1}(B))=\sum_{k=0}^{\infty}\mu(A\cap F^{-k-n-1}(B)\cap D_k)$$

We now apply Proposition 2 to each term of the sum and we get

$$\begin{aligned}\mu(A\cap F^{-\tau-n-1}(B))&\leqslant(1+C\theta^n)\sum_{k=0}^{\infty}\mu(A\cap\cap D_k)\,\mu(B)\\&=(1+C\theta^n)\,\mu(A)\,\mu(B)\end{aligned}$$

The lower bound is obtained in a similar way and this proves the proposition.

Proposition 4. There is a positive constant $\bar{C}>1$ such that for any positive number a smaller than 1 we have

$$a/\bar{C}\leqslant\mu\{U>1/a\}\leqslant\bar{C}a$$

Proof. From our hypothesis on f it follows[2] that F satisfies the following uniform distortion property:

$$\sup_{x\in[1/2,1]}\frac{|F''(x)|}{F'^2(x)}<\infty$$

This implies that the density of the invariant probability measure μ is bounded as well as its inverse. On the other hand, it follows from Lemma A2 in ref. 4 that the set $\{U > 1/a\}$ is an interval with length of order a and this finishes the proof.

Remark. This immediately implies that the positive function U is not integrable with respect to μ. This is reminiscent of the null recurrent situation for Markov chains. In particular this implies that the nontrivial invariant measure of the map f is not finite. We refer to ref. 6 for the piecewise affine case, where the analogy with null recurrent Markov chains is explicit.

4. *A PRIORI* ESTIMATES

We first introduce some additional notation. For x in the interval $]1/2, 1]$, we will denote by $\tau_a^k(x)$ the time of the kth visit to the interval $U^{-1}([0, a])$ of the orbit of x under the induced map F. In other words,

$$\tau_a^1(x) = \inf\{j \geqslant 0 \colon U \circ F^j(x) \geqslant 1/a\}$$

and recursively

$$\tau_a^{k+1}(x) = \inf\{j > \tau_a^k(x) \colon U \circ F^j(x) \geqslant 1/a\}$$

We remark that all these functions are stopping time; moreover, the ergodicity of μ implies that they are almost surely finite.

With this definition we can rewrite T_a^k as

$$T_a^k = 1 + \sum_{j=0}^{\tau_a^k - 1} U \circ F^j \tag{4.1}$$

with the usual convention that the sum is zero if the range is empty.

We also define the integer-valued random variable N_t, which counts the number of returns of the path starting at x to the interval $[1/2, 1]$ until time t (a positive real number). It is given by

$$N_t(x) = \sup\left\{j \geqslant 0;\ \sum_{l=0}^{j} U \circ F^l(x) \leqslant t\right\}$$

We define the time scale β_a by

$$\beta_a = \min\{n \in \mathbb{N} \colon \mu\{T_a^1 \geqslant n\} \leqslant e^{-1}\} \tag{4.2}$$

where μ is the absolutely continuous probability invariant under F. The ergodicity of μ imply that β_a is finite and well defined.

The first step in the proof of the main Theorem is the following.

Theorem 5. For any positive real numbers t, the following limit holds:

$$\lim_{\substack{a \to 0^+ \\ a \in I}} \mu\{\beta_a^{-1} T_a^1 > t\} = e^{-t}$$

Moreover,

$$\lim_{\substack{a \to 0^+ \\ a \in I}} \beta_a^{-1} \int T_a^1 \, d\mu = 1$$

The main idea of the proof is the following. The time needed to perform the first visit to the interval $[0, a]$ is much larger than the typical mixing time. This will imply the factorization property announced in Theorem 5. Of course, this will be exact only in an asymptotic sense and the result will be nontrivial if one uses a suitable time scale.

In order to complete this program we first derive an *a priori* bound for T_a^1.

Proposition 6. There exists an increasing positive function L defined on I such that

$$\lim_{a \to 0} aL(a) = +\infty$$

and

$$\lim_{a \to 0} \sup_{s \geq 0} \mu\{s \leq T_a^1 < s + L(a)\} = 0$$

In order to prove this we first need two auxiliary lemmata.

Lemma 7. There exists an increasing integer-valued function l defined on the integers such that

$$\lim_{r \to \infty} \frac{l(r)}{r} = \infty$$

and

$$\lim_{r \to \infty} \mu\{N_{2l(r)} \geq r - 1\} = 0$$

Proof. By definition we have

$$\{N_{2l(r)} \geq r - 1\} = \{U + U \circ F + \cdots + U \circ F^{r-1} \leq 2l(r)\}$$

We now consider the Laplace transform of the random variable

$$W_r = \sum_{i=0}^{r-1} U \circ F^i$$

From Markov's inequality we have for $t \in [0, 1[$

$$\mu(W_r \leqslant l) \leqslant t^{-l} \int t^{W_r}\, d\mu$$

Let m and s be two integers chosen below such that $ms \leqslant r$. Let $\tilde{W}_r$ be the function defined by

$$\tilde{W}_r = \sum_{l=0}^{s-1} U \circ F^{lm}$$

We have obviously

$$W_r \geqslant \sum_{j=0}^{m-1} \tilde{W}_r \circ F^j$$

Using recursively Hölder's inequality, we obtain

$$\int t^{W_r}\, d\mu \leqslant \left(\int t^{m\tilde{W}_r}\, d\mu \right)^{1/m} \left(\int t^{m/(m-1)\sum_{j=1}^{m-1} \tilde{W}_r \circ F^j}\, d\mu \right)^{m-1/m} \leqslant \cdots \leqslant \int t^{m\tilde{W}_r}\, d\mu$$

Using the mixing property expressed in Proposition 2, in the last term we obtain the upper bound

$$\int t^{W_r}\, d\mu \leqslant \left(C\theta^{m-1} + \int t^{mU}\, d\mu \right)^{s-1} \leqslant e^{(s-1)[-g(t^m) + C\theta^{m-1}]}$$

where $g(t)$ is the function

$$g(t) = 1 - \int t^U\, d\mu$$

As we have already remarked, Proposition 4 implies the nonintegrability of the function U. Therefore the function $g(t)$ satisfies $\lim_{t \to 1_-} g(t) = 0$ and $\lim_{t \to 1_-} g(t)/(1-t) = +\infty$.

This suggests to call $t = 1 - u$, with $u > 0$ small, and to define $\sigma(u) = g(1-u)/u$. A standard straightforward computation leads to the equality

$$\int t^U\, d\mu = 1 + (1 - t^{-1}) \sum_{n=1}^{\infty} \mu\{U \geqslant n\}\, t^n$$

and therefore to the formula

$$\sigma(u) = \sum_{n=1}^{\infty} \mu\{U \geqslant n\}(1-u)^{n-1}$$

From this it is obvious that σ is a decreasing function of u.

Using the inequalities

$$1 - u \geqslant e^{-u-u^2} \qquad \text{and} \qquad (1-u)^m \leqslant 1 - mu + m^2u^2$$

as well as the monotonicity of g, we get

$$\mu(W_r \leqslant 1) \leqslant \exp\{u[l + lu - \sigma(mu - m^2u^2)\, m(s-1)(1-mu)] + C(s-1)\theta^m\}$$

Let us take $u = 1/r$, $m = [(\log r)^2]$, $s = [r/(\log r)^2]$, and finally

$$l(r) = [r\{\sigma((\log r)^2/r)\}^{1/2}]$$

where $[\cdot]$ denotes the integer part.

By a direct elementary computation, one can check that with this choice the result follows.

We remark that it is easy also to prove that the positive ratio $-\sigma(u)/\log u$ is bounded away from zero and infinity when $u \to 0^+$.

Lemma 8. Let C be the constant which appears in Proposition 2; then:

(i) $C^{-1} \leqslant \mu\{U \geqslant 1/a\} \int \tau_a^1 \, d\mu \leqslant C$.

(ii) Moreover, for any positive integer k, $\mu\{\tau_a^1 \leqslant k\} \leqslant k\mu\{U \geqslant 1/a\}$.

Proof. We first observe that

$$1_{\{\tau_a^1 < \infty\}} = 1_{\{U \geqslant 1/a\}} + \sum_{n=1}^{\infty} 1_{\{U \geqslant 1/a\}} \circ F^n \prod_{l=0}^{n-1} 1_{\{U < 1/a\}} \circ F^l$$

Since $\mu\{\tau_a^1 < \infty\} = 1$, using the uniform mixing property given by Proposition 2 as well as the invariance of μ, we get

$$C^{-1} \leqslant \mu\{U \geqslant 1/a\} \sum_{n=0}^{\infty} \mu\{\tau_a^1 \geqslant n\} \leqslant C$$

From this we obtain (i) immediately.

Inequality (ii) is also a standard consequence of the invariance of μ. Indeed

$$1_{\{\tau_a^1 \leqslant k\}} = 1_{\{U \geqslant 1/a\}} + \sum_{n=1}^{k} 1_{\{U \geqslant 1/a\}} \circ F^n \prod_{l=0}^{n-1} 1_{\{U < 1/a\}} \circ F^l$$

$$\leqslant 1_{\{U \geqslant 1/a\}} + \sum_{n=1}^{k} 1_{\{U \geqslant 1/a\}} \circ F^n$$

Integrating with respect to μ, the result follows.

We remark that using Markov's inequality and the above estimate (i), we get the lower bound

$$\mu\{\tau_a^1 \geqslant k\} \leqslant \frac{C}{k\mu\{U \geqslant 1/a\}}$$

for any positive integer k.

Proof of Proposition 6. For any fixed a and $s > 1/a$, we define a partition (A_k) by

$$A_k = \{x: N_{s-1/a}(x) = k\}$$

We denote by $r(a)$ a function to be defined below. Using the partition (A_k), we get

$$\mu\{s \leqslant T_a^1 < s + l(r(a))\}$$
$$\leqslant \sum_k \mu(A_k \cap F^{k+2}(B \cap D)) + \sum_k \mu(A_k \cap F^{k+2}(B \cap D^c))$$

where

$$B = \bigcup_{i=0}^{\infty} (\{\tau_a^1 \leqslant i\} \cap \{N_{l(r(a))+1/a} \geqslant i\})$$

and

$$D = \{N_{l(r(a))+1/a \leqslant r(a)}\}$$

We now observe that

$$B \cap D \subset \{\tau_a^1 \geqslant r(a)\}$$

Therefore using Proposition 2, we get

$$\sum_k \mu(A_k \cap F^{k+2}(B \cap D)) \leqslant C \sum_k \mu(A_k)\, \mu(\{\tau_a^1 \geqslant r(a)\})$$
$$= C\mu\{\tau_a^1 \geqslant r(a)\}$$

We also have from Proposition 2

$$\sum_k \mu(A_k \cap F^{k+2}(B \cap D^c)) \leqslant C\mu(D^c)$$

This finally gives the upper bound

$$\mu\{s \leqslant T_a^1 < s + l(r(a))\} \leqslant \mu\{U \geqslant 1/a\}\, r(a) + \mu\{N_{2l(r(a))} \geqslant r(a)\}$$

We now choose the function $r(a)$ in such a way that both probabilities in the above upper bound vanish to 0 when $a \to 0^+$.

We define the function

$$\omega(z) = \inf_{y \geqslant z} l(y)/y$$

This is a nondecreasing function diverging with z. Therefore there exists a nonincreasing function $r(a)$ satisfying

$$a = \frac{1}{r(a)} \left\{ \frac{1}{\omega(r(a))} \right\}^{1/2}$$

Note that $r(a) \to \infty$, and $ar(a) \to 0$ when $a \to 0^+$. By Proposition 4 we have $\mu\{U \geqslant 1/a\} \leqslant \mathcal{O}(1)a$. Now the result follows from Lemmata 7 and 8 by choosing $L(a) = l(r(a))$.

Corollary 9. We have

$$\lim_{a \to \infty} a\beta_a = +\infty$$

Proof. We first remark that by definition

$$\mu\{T_a^1 < \beta_a\} > 1 - e^{-1}$$

The second part of Proposition 6 implies that for a small enough, we have $L(a) \leqslant \beta_a$. The conclusion follows from the first part of this proposition.

5. PROOF OF THEOREM 5

Let $G_a(t)$ be the function defined by

$$G_a(t) = \mu\{\beta_a^{-1} T_a^1 \geqslant t\}$$

We are going to prove that for any pair of positive real numbers s and t, the following asymptotic factorization property holds:

$$\lim_{a \to 0^+} (G_a(t+s) - G_a(t)\, G_a(s)) = 0$$

Among distribution functions, this factorization property is satisfied only by exponential functions or by the trivial constant functions 0 or 1.

We introduce again a partition (A_k), which is now defined by

$$A_k = \{x\colon N_{\beta_a t}(x) = k\}$$

Let $m = [\{aL(a)\}^{1/2}]$; this is a positive function of a diverging when a tends to 0 and such that $aL(a)/m$ also diverges. It will be useful to introduce also the set B given by

$$B = \{\tau_a^1 \leqslant m+1\}$$

It is easy to verify that for any integer k

$$(\{T_a^1 > \beta_a t\} \cap \{T_a^1 \circ F^{m+k} > \beta_a s - m\} \cap F^k(B^c)) \cap A_k$$
$$\subset \{T_a^1 > \beta_a(s+t)\} \cap A_k$$

and also

$$\{T_a^1 > \beta_a(s+t)\} \cap A_k$$
$$\subset (\{T_a^1 > \beta_a t\} \cap \{T_a^1 \circ F^{m+k} > \beta_a s - (m+1)/a\} \cup \{\tau_a^1 \circ F^k \leqslant m+1\}) \cap A_k$$

Therefore, summing the measures of these sets over k, we get

$$\sum_k (\mu(\{T_a^1 > \beta_a t\} \cap A_k \cap \{T_a^1 \circ F^{m+k} > \beta_a s - m\}) - \mu(B \cap A_k)) \leqslant G_a(t+s)$$

and also

$$G_a(t+s)$$
$$\leqslant \sum_k \mu(\{T_a^1 > \beta_a t\} \cap A_k \cap \{T_a^1 \circ F^{m+k} > \beta_a s - (m+1)/a\} \cup \{\tau_a^1 \circ F^k \leqslant m+1\})$$

We remark that $B \subset \{T_a^1 \leqslant (m+1)/a\}$. Using the decay of correlations given by Proposition 3, we obtain

$$\sum_k \mu(\{T_a^1 > \beta_a t\} \cap A_k \cap \{T_a^1 \circ F^{m+k} > \beta_a s - m\})$$
$$\geqslant (1 - C\theta^{m-1})\, \mu\{T_a^1 > \beta_a s - m\} \sum_k \mu(\{T_a^1 > \beta_a t\} \cap A_k)$$
$$\geqslant (1 - C\theta^{m-1})\, \mu\{T_a^1 > \beta_a s\}\, \mu\{T_a^1 > \beta_a t\}$$

Therefore by Proposition 6 we have

$$\lim_{a \to 0^+} \mu(B) = 0$$

Since m diverges when a tends to 0, this implies that

$$\lim_{a \to 0^+} (G_a(t+s) - G_a(t)\, G_a(s)) \geqslant 0$$

Similarly, we obtain the upper bound

$$G_a(t+s) \leqslant (1 + C\theta^{m-1})\, G_a(t)\, \mu\{T_a^1 > \beta_a s - (m+1)/a\}$$

Since $(m+1)/aL(a)$ tends to 0 with a, Proposition 6 implies

$$\lim_{a \to 0^+} (\mu\{T_a^1 > \beta_a s - (m+1)/a\} - G_a(s)) = 0$$

which concludes the proof of the factorization property.

Now we must prove that the functions G_a converge to a nontrivial limit. We first remark that by definition we have

$$\mu\{T_a^1 \geqslant \beta_a - 1\} > e^{-1} \geqslant \mu\{T_a^1 \geqslant \beta_a\}$$

Therefore it follows from Proposition 6 that

$$\lim_{a \to 0^+} G_a(1) = e^{-1}$$

Using the factorization property, we conclude that for any positive rational number t, we have

$$\lim_{a \to 0^+} G_a(t) = e^{-t}$$

Since the exponential is a continuous function, this concludes the proof of the convergence in law of $\beta_a^{-1} T_a^1$.

We now turn to the proof of the second part of Theorem 5. We first remark that by an integration by parts followed by a change of variables we get

$$\beta_a^{-1} \int T_a^1\, d\mu = \int_0^\infty G_a(t)\, dt$$

Since $G_a(t)$ converges to e^{-t}, the result will follow if we can use the Lebesgue dominated convergence theorem. We have proven above that for any $t > 1$

$$G_a(t) \leqslant G_a(t-1)(1 + C\theta^{m-1})\, \mu\{T_a^1 > \beta_a - (m+1)/a\}$$

Proposition 6 implies that when a tends to 0, the quantity

$$\mu\{T_a^1 > \beta_a - (m+1)/a\}$$

converges to e^{-1}. Since m diverges with a, we conclude that for a small enough we have

$$G_a(t) \leqslant e^{-([t]-1)/2}$$

and this concludes the proof of the theorem.

6. PROOF OF THE MAIN THEOREM

We first consider the case in which the starting point is chosen according to the invariant measure μ.

Theorem 10. For any positive integer n and any sequence of positive real numbers $s_1, s_2,\ldots$ the following holds:

$$\lim_{a \to 0^+} \mu\{T_a^1 > \beta_a s_1, T_a^2 - T_a^1 > \beta_a s_2,\ldots, T_a^n - T_a^{n-1} > \beta_a s_n\} = \prod_{j=1}^{n} e^{-s_j}$$

Proof. The proof is by induction. The case $n = 1$ was proven in Theorem 5. The proof of the induction step follows essentially the same scheme as the proof of Theorem 5. The new point is that now we must control the length of the first return time to the interval $[1/2, 1]$ after each entrance in the interval $]0, a]$.

Let us call D_n the set

$$D_n = \{T_a^1 > \beta_a s_1, T_a^2 - T_a^1 > \beta_a s_2,\ldots, T_a^n - T_a^{n-1} > \beta_a s_n\}$$

and let us assume that the result holds up to the integer n. We are now going to prove it for $n+1$.

We observe that

$$D_{n+1} = (D_n \cap \{T_a^1 \circ F^{\tau_a^n + 1} + U \circ F^{\tau_a^n} > \beta_a s_{n+1}\})$$

In order to control the probability of this set, we will find a larger and a small set with comparable measures. The set D_{n+1} is contained in the union of the two sets

$$D_n \cap \{U \circ F^{\tau_a^n} \leqslant m/a\} \cap \{T_a^1 \circ F^{\tau_a^n + 1} > \beta_a s_{n+1} - m/a\}$$

and

$$\{U \circ F^{\tau_a^n} > m/a\}$$

Since

$$\{\tau_a^1 \circ F^{\tau_a^n + 1} < m\} \subset \{T_a^1 \circ F^{\tau_a^n + 1} < m/a\}$$

using

$$m/a < \beta_a s_{n+1} - m/a$$

for a small enough, we conclude that the set

$$D_n \cap \{U \circ F^{\tau_a^n} \leqslant m/a\} \cap \{T_a^1 \circ F^{\tau_a^n + 1} > \beta_a s_{n+1} - m/a\}$$

is contained in the set

$$D_n \cap \{U \circ F^{\tau_a^n} \leqslant m/a\} \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - 2m/a\}$$

We are now ready to use the mixing property to get an upper bound for the measure of the above set. Using Corollary 3, we get

$$\begin{aligned}
&\mu((D_n \cap \{U \circ F^{\tau_a^n} \leqslant m/a\} \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - 2m/a\})) \\
&\quad \leqslant (1 + C\theta^m)\, \mu(D_n \cap \{U \circ F^{\tau_a^n} \leqslant m/a\})\, \mu(\{T_a^1 > \beta_a s_{n+1} - 2m/a\}) \\
&\quad \leqslant (1 + C\theta^m)\, \mu(D_n)\, \mu(\{T_a^1 > \beta_a s_{n+1} - 2m/a\})
\end{aligned}$$

Using the induction hypothesis, Theorem 5, and Proposition 6, we conclude that the above number converges to

$$\prod_{j=1}^{n} e^{-s_j}$$

We now prove that the remaining term in the upper bound

$$\mu\left(\bigcup_k (A_k \cap (\{U \circ F^{\tau_a^n} > m/a\})\right)$$

converges to 0. We do that by introducing the partition (A_k) defined in the following way:

$$A_k = \{\tau_a^n = k\}$$

Since

$$\begin{aligned}
&A_k \cap \{U \circ F^{\tau_a^n} > m/a\} \\
&\quad = A_k \cap \{U \circ F^k > m/a\} \subset \{\tau_a^n \geqslant k - 1\} \cap \{U \circ F^k > m/a\}
\end{aligned}$$

we can use Proposition 2 to conclude that

$$\begin{aligned}
\mu\left(\bigcup_k (A_k \cap (\{U \circ F^k > m/a\})\right) &\leqslant C\mu\{U > m/a\} \sum_{k \geqslant n} \mu\{\tau_a^n > k - 1\} \\
&\leqslant C\mu\{U > m/a\} \int \tau_a^n \, d\mu
\end{aligned}$$

Using

$$\int \tau_a^n \, d\mu = n \int \tau_a^1 \, d\mu$$

and the first part of Lemma 8, we finally get

$$\mu\left(\bigcup_k (A_k \cap \left(\left\{U \circ F^k > \frac{m}{a}\right\}\right)\right) \leqslant nC \frac{\mu\{U > m/a\}}{\mu\{U > 1/a\}}$$

This last quantity tends to zero by Proposition 4.

We now prove a lower bound for the measure of D_{n+1}. By definition, $U \circ F^{\tau_a^n} \geqslant 1/a$. Therefore the set D_{n+1} contains the set

$$D_n \cap \{T_a^1 \circ F^{\tau_a^n + 1} > \beta_a s_{n+1} - 1/a\}$$

We will consider the even smaller set

$$D_n \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - m/a\} \cap \{\tau_a^1 \circ F^{\tau_a^n + 1} > m\}$$

We now have the lower bound

$$\begin{aligned} &\mu((D_n \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - m/a\} \cap \{\tau_a^1 \circ F^{\tau_a^n + 1} > m\})) \\ &\quad \geqslant \mu((D_n \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - m/a\})) - \mu((\{\tau_a^1 \circ F^{\tau_a^n + 1} \leqslant m\})) \end{aligned}$$

Using in both terms the mixing property given in Corollary 3, we get

$$\begin{aligned} &\mu((D_n \cap \{T_a^1 \circ F^{\tau_a^n + m} > \beta_a s_{n+1} - m/a\})) \\ &\quad \geqslant (1 - C\theta^m)\, \mu(D_n)\, \mu(\{T_a^1 > \beta_a s_{n+1} - m/a\}) \end{aligned}$$

and

$$\mu((\bigcap \{\tau_a^1 \circ F^{\tau_a^n + 1} \leqslant m\})) \leqslant C\mu(\{\tau_a^1 \leqslant m\})$$

As before, the first quantity converges to

$$\prod_{j=0}^{n+1} e^{-s_j}$$

and the second one converges to zero by the second part of Lemma 8. This concludes the proof of Theorem 10.

We now conclude the proof of the Main Theorem. In order to do this, it is enough to extend Theorem 10 to the nonequilibrium situation. In other words, for any positive function g of bounded variation and integral 1 we have to prove that for any integer n

$$\lim_{a \to 0^+} \left(\mu(D_n) - \int \chi_{D_n}(x)\, g(x)\, dx \right) = 0$$

where the set D_n is given by

$$D_n = \{T_a^1 > \beta_a s_1, T_a^2 - T_a^1 > \beta_a s_2, \dots, T_a^n - T_a^{n-1} > \beta_a s_n\}$$

We first remark that

$$\int \chi_{D_n}(x)\, g(x)\, dx - \int \chi_{D_n}(x)\, \chi_{\{T_a^1 \geqslant L(a)\}}\, g(x)\, dx$$

is positive and bounded above by

$$\int \chi_{\{T_a^1 < L(a)\}}\, g(x)\, dx \leqslant \|g\|_\infty \|1/h\|_\infty \int \chi_{\{T_a^1 \geqslant L(a)\}} h(x)\, dx$$

By Proposition 6 this last quantity converges to 0 with a. Therefore it is enough to prove that

$$\lim_{a \to 0^+} \left(\mu(D_n) - \int \chi_{D_n}(x)\, \chi_{\{T_a^1 \geqslant L(a)\}}\, g(x)\, dx \right) = 0$$

The proof now follows exactly the pattern of the proof of Theorem 5. We start by indicating how to treat the case $n = 1$ with $s_1 = s$. We introduce again a partition (A_k) which is now defined by

$$A_k = \{x\colon N_{L(a)}(x) = k\}$$

It will be useful to introduce also the set B given by

$$B = \{\tau_a^1 \leqslant m + 1\}$$

where $m = [\{aL(a)\}^{1/2}]$.

It is easy to verify that for any integer k

$$(\{T_a^1 > L(a)\} \cap \{T_a^1 \circ F^{m+k} > \beta_a s - m\} \cap B^c) \cap A_k \subset \{T_a^1 > \beta_a s\} \cap A_k$$

and also

$$\{T_a^1 > \beta_a s\} \cap A_k \subset (\{T_a^1 > L(a)\} \cap \{T_a^1 \circ F^{m+k} > \beta_a s - (m+1)/a\}) \cap A_k$$

Now we follow step by step what we did to prove Theorem 5. The procedure is the same for a general integer n. This concludes the proof of the Main Theorem.

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